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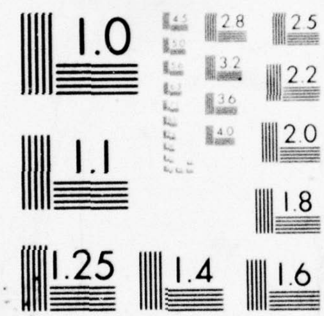

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(6) EIGENVALUE PROBLEM FOR A CLASS OF CYCLICALLY MAXIMAL MONOTONE OPERATORS

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EIGENVALUE PROBLEM FOR A CLASS OF CYCLICALLY MAXIMAL MONOTONE OPERATORS†

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ABSTRACT

In this paper, we prove Lyusternik-Schnirelmann type theorems for convex functions in a Hilbert space. We consider the problems $\partial\varphi(u) \ni \lambda u$ (and $(\partial\varphi)^{-1}(u) \ni \lambda u$) where $\partial\varphi$ is the subdifferential of an even convex function φ in a Hilbert space. We give conditions on φ for there to exist infinitely many distinct pairs of solutions having a prescribed norm.

AMS(MOS) Subject Classification - 47H05, 58E15.

Key Words - Nonlinear eigenvalue problem, subdifferential of a convex function, genus, infmax, supmin, Yosida approximation.

Work Unit Number 1 - Applied Analysis

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Eigenvalue problem for a class of cyclically maximal monotone operators†

Philippe Clément

1. Introduction

The class of maximal monotone operators in a Hilbert space plays an important role in nonlinear functional analysis since it can be identified with the class of generators of nonlinear contraction semigroups in a Hilbert space [1]. Therefore it is natural to consider corresponding eigenvalue problems for it. As in the linear case, the study of such problems is considerably simplified if a variational structure is present. In terms of a maximal monotone operator A in a Hilbert space H, this means that A is the subdifferential $\partial\varphi$ of a lower-semi continuous convex function $\varphi: H \to]-\infty, \infty]$, i.e. $y \in \partial\varphi(x)$ iff $\varphi(z) - \varphi(x) \geq (y, z - x)$ for all z in H. In this paper, we are concerned with the problem

$$\text{(i)} \qquad \partial\varphi(u) \ni \mu u$$

where the norm of u, $|u|$ is prescribed. As in the linear case, we assume that $0 \in \partial\varphi(0)$. Therefore without loss of generality we can assume that φ takes its values in $[0, \infty]$ and $\varphi(0) = 0$.

It is known [see for example Theorem 2.10 of [2]] that if H is a real infinite dimensional Hilbert space, $\varphi \in C^1(H, \mathbb{R})$, is even, bounded from below, and satisfies the Palais-Smale condition for some sphere (i.e. there is an $R > 0$

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such that every sequence (u_n), with $|u_n| = R$, along which $|\varphi(u_n)|$ is bounded and $\varphi'(u_n) - (\varphi'(u_n), u_n)u_n$ converges to 0, where $\varphi'(u)$ denotes the Fréchet derivative of φ at u, possesses a convergent subsequence, then the equation $\varphi'(u) = \mu u$, $|u| = R$, has infinitely many distinct pairs of solutions. In our case, we prove that if the effective domain of φ, $D(\varphi) := \{x \in H \mid \varphi(x) < \infty\}$ is dense in H, φ is even and the condition: $\{x \in H \mid \varphi(x) \leq \lambda\}$ is compact for all $\lambda \geq 0$, is satisfied, then Equation (i) possesses infinitely many distinct pairs of solutions.

We also consider a "dual" problem, where the existence of solutions of

(ii) $\qquad (\partial\varphi)^{-1}(v) \ni \nu v, \quad |v| = R > 0$

is proved. In this case the same compactness condition on φ as in the first problem is employed but instead of requiring that $D(\varphi)$ is dense in H, we assume that φ satisfies a coercivity condition. The same technique is used to prove the existence of positive solutions when the Hilbert space is equipped with a cone P and $D(\varphi)$ is dense in P. We conclude with a simple application to a second-order nonlinear elliptic partial differential equation. Further applications can be obtained by using known results establishing the maximal monotonicity of specific operators.

Concerning the proofs, the eigenvalue problem for the Yosida approximation of φ (or φ^* the conjugate function of φ) is studied first. Here we can use the techniques developed initially by Lyusternik, Schnirelmann [3], and Krasnoselski [4]. Then we get the results for φ by passing to the limit employing closedness properties of maximal monotone operators. [See [1] and the references of [1]]. Finally a lower bound on the number of solutions is obtained by using arguments related to those of [4]. See also the references of [2].

Different results concerning eigenvalue problems for maximal monotone operators can be found in [5], [6] where the existence of solutions is based on a result of Rabinowitz [7] about global branches of solutions emanating from a bifurcation point which generalizes a previous result of Krasnoselski [4]. In our context, it can happen that no bifurcation occurs.

The author would like to express his gratitude to Professor P. H. Rabinowitz for his suggestions and helpful assistance.

2. Results

Let H be a real infinite dimensional separable Hilbert space, with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$.

Theorem 1. Let $\varphi: H \to [0, \infty]$ be convex and even, satisfying:

$$\varphi(0) = 0 , \tag{1}$$

$$\{x \in H \mid \varphi(x) \leq \lambda\} \text{ is compact for all } \lambda \geq 0 , \tag{2}$$

$$D(\varphi) \text{ is dense in } H . \tag{3}$$

Then for all $R > 0$, there exists a sequence $(\mu_k, u_k) \in \mathbb{R}^+ \times H$, $k \in \mathbb{N}$ such that:

a) $|u_k| = R$

b) $\partial\varphi(u_k) \ni \mu_k u_k, \quad \mu_k \geq 0$

c) $\sup\limits_{k \in \mathbb{N}} \varphi(u_k) = +\infty$.

<u>Theorem 2.</u> Let $\varphi: H \to [0, \infty]$ be convex and even, satisfying:

$$\varphi(0) = 0, \quad 0 = \partial\varphi(0) . \tag{1}$$

$$\{x \in H \mid \varphi(x) \leq \lambda\} \text{ is compact for all } \lambda \geq 0 , \tag{2}$$

$$\lim_{\substack{|x| \to \infty \\ x \in D(\varphi)}} \frac{\varphi(x)}{|x|} = \infty . \tag{3}$$

Then for all $R > 0$, there exists a sequence $(\nu_k, v_k) \in \mathbb{R}^+ \times H$, $k \in \mathbb{N}$ such that:

a) $|v_k| = R$

b) $\partial\varphi^{-1}(v_k) \ni \nu_k v_k, \quad \nu_k > 0$.

c) $\inf\limits_{k \in \mathbb{N}} \varphi^*(v_k) = 0$,

where $\varphi^*(x) := \sup\limits_{y \in H} \{(x, y) - \varphi(y)\}$ is the conjugate function of φ .

Let $P \subseteq H$ be a closed, convex cone. (In particular if $P \cap -P = \{0\}$, P is a cone of "positive" functions).

<u>Theorem 3.</u> Let $\varphi: H \to [0, \infty]$ be convex and satisfy:

$$\varphi(0) = 0 \ , \tag{1}$$

$$\{x \in H \mid \varphi(x) \leq \lambda\} \text{ is compact for all } \lambda \geq 0 \ , \tag{2}$$

$$\overline{D(\varphi)} = P, \quad \text{where } \overline{D(\varphi)} \text{ denotes the closure of } D(\varphi) \ . \tag{3}$$

Then for all $R > 0$, there exists $(\mu, u) \in \mathbb{R}^+ \times P$ such that

a) $|u| = R$

b) $\partial\varphi(u) \ni \mu u \qquad \mu \geq 0$

c) $\varphi(u) = \inf\limits_{|v| = R} \varphi(v)$.

Theorem 4.

Let φ: $H \to \mathbb{R}$ be convex and weakly continuous. Then for all $R > 0$, there exists $\lambda \geq 0$, $u \in H$ such that

a) $|u| = R$

b) $\partial\varphi(u) \ni \lambda u$

c) $\varphi(u) = \sup\limits_{|v|=R} \varphi(v)$.

Remark. In the theorems 1 and 2, the conditions $\sup\limits_{k\in \mathbb{N}} \varphi(u_k) = \infty$ and $\inf\limits_{k\in\mathbb{N}} \varphi^*(v_k) = 0$ imply that the equations $\partial\varphi(u) \ni \mu u$, $|u| = R$ and $(\partial\varphi)^{-1}(v) \ni \nu v$, $|v| = R$ possess infinitely many pairs of distinct solutions. Clearly by the oddness of $\partial\varphi(\partial\varphi^{-1})$, if (μ, u) is a solution, then $(\mu, -u)$ is another one. Moreover, in the first case, since u_k belongs to the domain of $\partial\varphi$, $\varphi(u_k) < \infty$, hence the number of distinct solutions is infinite. Similarly, in the second case,

$\varphi^*(v_k) > 0$, otherwise, by definition $\sup_{y \in H}\{(v_k, y) - \varphi(y)\} = 0$, hence $\varphi(y) \geq (v_k, y)$ for all $y \in H$. $\varphi(0) = 0$ implies that $v_k \in \partial\varphi(0)$ and the assumption (1), $v_k = 0$. which is impossible since $|v_k| = R > 0$. Therefore the number of distinct solutions is infinite.

3. <u>Auxiliary lemmas and notations</u>

Let H a real infinite dimensional separable Hilbert space. For $R > 0$, B_R is the open ball of radius R, $\overline{B}_R$ its closure and ∂B_R its boundary. $x_n \to x$ denotes a strongly convergent sequence and $x_n \rightharpoonup x$ a weakly convergent sequence. For $f \in C^1(H, \mathbb{R})$, $f'(x)$ is the Fréchet derivative of f at x. $f|_{\partial B_R}$ denotes the restriction of f to ∂B_R. $x \in \partial B_R$ is called a critical point of $f|_{\partial B_R}$ if $f'(x) - (f'(x), x) R^{-2} x = 0$. $a \in \mathbb{R}$ is a critical value of $f|_{\partial B_R}$ if $a = f(x)$ for some critical point x of $f|_{\partial B_R}$.

For $R > 0$, $b \in \mathbb{R}$, if every sequence (x_n), $x_n \in \partial B_R$, along which $f(x_n) \to b$ and $|f'(x_n) - (f'(x_n), x_n)R^{-2} x_n| \to 0$ possesses a strongly convergent subsequence, we shall say that $f|_{\partial B_R}$ satisfies the Palais-Smale (P. S.) condition at b [2].

It is known that the notion of genus is useful for the characterization of critical values. Let us recall some facts about the genus of a set. Let $\Sigma(H) := \{A \subseteq H - \{0\} \mid A \text{ closed, symmetric}\}$. For $A \in \Sigma(H)$, let $\gamma(A) := \operatorname{Inf}\{n \in \mathbb{N} \mid$ there exists $g: A \to \mathbb{R}^n - \{0\}$ which is continuous and odd$\}$ with the convention $\operatorname{Inf} \emptyset = +\infty$. $\gamma(A)$ is called the genus of A. [2], [9]. It follows immediately from the definition that if $\gamma(A) = k$, $B \in \Sigma(H)$ and there exists $g: A \to B$ continuous and odd, then $\gamma(B) \geq k$. In particular, if $A \subseteq B$, $\gamma(A) \leq \gamma(B)$. We shall use the fact that if $A \in \Sigma(H)$, A compact, then $\gamma(A) < \infty$, and if $A \in \Sigma(H)$ is homeomorphic to a k dimensional sphere S^{k-1} by an odd homeomorphism, $\gamma(A) = k$. [2], [4].

For $R > 0$, $k \in \mathbb{N}$, we define $\gamma_k^R := \{A \subseteq \partial B_R \mid A$ compact, symmetric, $\gamma(A) \geq k\}$. Clearly $\gamma_k^R \neq \emptyset$ and $\gamma_{k+1}^R \subseteq \gamma_k^R$. When the context is clear, we shall omit the subscript R. Finally, we recall that if $f: H \to \mathbb{R}$ is weakly continuous $f \geq 0$, $f(x) = 0$ if and only if $x = 0$, then $c_k^R := \sup_{\Gamma \in \gamma_k^R} \inf_{x \in \Gamma} f(x)$ satisfies: i) $0 < c_k^R < \infty$ and ii) $\inf_{k \in \mathbb{N}} c_k^R = 0$. [2], [4].

We are now ready to state the first lemma.

<u>Lemma 1</u>. Let $f \in C^1(H, \mathbb{R})$ with $|f'(x) - f'(y)| \leq M|x-y|$ for some $M > 0$. Let $R > 0$ and $b = \inf_{|v|=R} f(v)$.

If f is even, for $k \in \mathbb{N}$, let $b_k := \inf_{\Gamma \in \gamma_k^R} \sup_{x \in \Gamma} f(x)$ and $c_k := \sup_{\Gamma \in \gamma_k^R} \inf_{x \in \Gamma} f(x)$.

Assume that $f|_{\partial B_R}$ satisfies the (P.S) condition at b (resp. b_k, c_k), then b(resp. b_k, c_k) is a critical value of $f|_{\partial B_R}$.

<u>Remark</u>. This lemma can be deduced from the results of [2], [4].

For the sake of completeness, we shall give a direct proof here.

<u>Proof of Lemma 1</u>

We consider the following associated differential equation:

$$\begin{cases} \dfrac{du}{dt} = -[f'(u) - (f'(u), u)R^{-2} u] \qquad t > 0 \\ u(0) = x \in H \ . \end{cases} \tag{3.1}$$

Since the right hand side is locally Lipschitz, for $x \in H$, (3.1) possesses a local solution on some interval $[0, t(x)[$. By taking the scalar product of (3.1) with u, we get, if $x \in \partial B_R$,

$$\begin{cases} \dfrac{1}{2} \dfrac{d}{dt} |u|^2 = -(f'(u), u) [1 - R^{-2} |u|^2] \\ |u(0)|^2 = R^2 \ . \end{cases}$$

By setting $v = |u|^2$ we get a linear equation for v, on $[0, t(x)[$. $v(t) = R^2$ is the unique solution, therefore $u(t)$ remains on ∂B_R. But on ∂B_R, the right hand side is globally Lipschitz and by a standard result we get:

i) $\forall x \in \partial B_R$, $\exists !$ $u(t, x)$ solution of (3.1) with $|u(t, x)|^2 = R^2$.

ii) $\forall t > 0$, $x \to u(t, x)$ is continuous, and odd if f is even.

By taking the scalar product of (3.1) with $\dot{u}$ we get: $|\dot{u}|^2 = -\frac{d}{dt} f(u(t, x))$ since $(u, \dot{u}) = 0$. Therefore $f(u(t))$ is decreasing. Since $|u(t)| = R$ and f maps bounded set into bounded sets, $f(u(t))$ is bounded. There exists $\alpha(x) \in \mathbb{R}$ such that: iii) $f(u(t, x)) \downarrow \alpha(x)$.

Consequently $\frac{d}{dt} f(u(t, x)) \to 0$, as $t \to \infty$. But $\frac{d}{dt} f(u(t, x)) = -|f'(u) - f'(u), u) R^{-2} u|^2$. Hence

iv) $|f'(u(t, x)) - (f'(u(t, x)), u(t, x))R^{-2} u(t, x)| \to 0$ as $t \to \infty$.

Now consider b (resp. b_k, c_k). These quantities are finite since f is bounded on ∂B_R. We claim that b (resp. b_k, c_k) is a critical value if for each $\varepsilon > 0$, there exists $x_\varepsilon \in \partial B_R$ such that $b \le f(u(t, x_\varepsilon)) \le b + \varepsilon$ for all $t > 0$. Indeed, if it is the case, for each $n \in \mathbb{N}$, we can find $x_n \in \partial B_R$ and $t_n > 0$, such that $b \le f(u(t_n, x_n)) \le b + \frac{1}{n}$ and $|f'(u(t_n, x_n)) - (f'(u(t_n, x_n)), u(t_n, x_n))R^{-2} u(t_n, x_n)| \le \frac{1}{n}$. Then if $y_n = u(t_n, x_n)$, $y_n \in \partial B_R$, $f(y_n) \to b$ and $|f'(y_n) - (f'(y_n), y_n)R^{-2} y_n| \to 0$. By (PS) at b, there exists a subsequence $y_{n_k} \to y$ and clearly y is a critical point of $f|_{\partial B_R}$ and $b = f(y)$ is a critical value.

First consider the case of $b = \inf\limits_{|v|=R} f(v)$. By definition, for each $\varepsilon > 0$, there exists $x_\varepsilon \in \partial B_R$ such that $f(x_\varepsilon) \le b + \varepsilon$. But $f(u(t, x_\varepsilon)) \le f(x_\varepsilon) \le b + \varepsilon$ for all $t > 0$ and $f(u(t, x_\varepsilon)) \ge b$ since $u(t, x_\varepsilon) \in \partial B_R$. This concludes the inf case.

Now let $b_k = \inf\limits_{\Gamma \in \gamma_k^R} \sup\limits_{x \in \Gamma} f(x)$. For each $\varepsilon > 0$, there exists $\Gamma_\varepsilon \in \gamma_k^R$ such that $\sup\limits_{x \in \Gamma_\varepsilon} f(x) \le b_k + \varepsilon$. Let $h_t(x) := u(t, x)$. Since h_t: $\partial B_R \to \partial B_R$ is continuous and odd, $h_t(\Gamma_\varepsilon) \in \gamma_k^R$. Therefore for all $t > 0$, $b_k \le \max\limits_{x \in h_t(\Gamma_\varepsilon)} f(x)$. Thus there exists $x(\varepsilon, t) \in \Gamma_\varepsilon$ such that $b_k \le f(u(t, x(\varepsilon, t)))$. Now choose a sequence $t_n \uparrow \infty$ and define $x_{\varepsilon, n} := x(\varepsilon, t_n)$. Since Γ_ε is compact, there exists $x_\varepsilon \in \Gamma_\varepsilon$ such that $x_{\varepsilon, n_j} \to x_\varepsilon$, $t_{n_j} \uparrow \infty$. We shall again denote this subsequence by $x_{\varepsilon, n}$ and t_n. For each $t > 0$, we have $u(t, x(\varepsilon, t_n)) \to u(t, x_\varepsilon)$. Hence $f(u(t, x(\varepsilon, t_n))) \to f(u(t, x_\varepsilon))$. We have

$b_k \leq f(u(t_n, x(\varepsilon, t_n))) \leq f(u(t, x(\varepsilon, t_n)))$ for $t_n \geq t$. Therefore $b_k \leq f(u(t, x_\varepsilon)) \leq f(x_\varepsilon)$ for each $t > 0$. But $x_\varepsilon \in \Gamma_\varepsilon$, thus $f(x_\varepsilon) \leq b_k + \varepsilon$. Thus, $b_k \leq f(u(t, x_\varepsilon)) \leq b_k + \varepsilon$ for all $t > 0$, and we are done. The case of $c_k := \sup_{\Gamma \in \gamma_k} \inf_{x \in \Gamma} f(x)$ follows from the preceding one, by observing that $-f$ satisfies the assumptions of the preceding case.

Now let us recall some definitions and results about convex functions in a Hilbert space. For references, see for example [1]. Let $\varphi: H \to [0, \infty]$ be convex, lower semi continuous with $\varphi(0) = 0$. Then $A = \partial\varphi$, the subdifferential of φ is $\neq \emptyset$ (since $0 \in \partial\varphi(0)$), maximal monotone and $J_\lambda^A := (I + \lambda A)^{-1}$ is a contraction, defined on all H for $\lambda > 0$. $A_\lambda := \frac{1}{\lambda}(I - J_\lambda^A)$ is called the Yosida approximation of A. We have $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$ and $\text{Int}\, D(\varphi) = \text{Int}\, D(\partial\varphi)$. $\varphi_\lambda(x) := \inf_{y \in H} \{\frac{1}{2\lambda}|x-y|^2 + \varphi(y)\} = \frac{\lambda}{2}|A_\lambda x|^2 + \varphi(J_\lambda x)$ is $C^1(H, \mathbb{R})$, $\varphi_\lambda \geq 0$, $\varphi_\lambda(0) = 0$ and $\varphi'_\lambda = A_\lambda$. A_λ is Lipschitz continuous with constant $\frac{1}{\lambda}$. $\varphi_\lambda(x) \uparrow \varphi(x)$ as $\lambda \downarrow 0$. We shall use the fact that if $\lim_{\substack{|x| \to \infty \\ x \in D(\varphi)}} \frac{\varphi(x)}{|x|} = \infty$, then $(\partial\varphi)^{-1}$ is defined everywhere and maps bounded sets into bounded sets.

The conjugate function of φ, $\varphi^*(x) := \sup_{y \in H} \{(x, y) - \varphi(y)\}$, is convex, lower semi continuous and ≥ 0, $\varphi^*(0) = 0$ and $(x, y) \leq \varphi(x) + \varphi^*(y)$ by definition. Also $(x, y) = \varphi(x) + \varphi^*(y)$ iff $x \in \partial\varphi^*(y)$ or $y \in \partial\varphi(x)$, since $\partial\varphi^* = (\partial\varphi)^{-1}$. We have $(\varphi^*)^* = \varphi$ and if $e(x) := \frac{1}{2}|x|^2$, $\varphi_\lambda = (\lambda e + \varphi^*)^*$. Let us prove this last identity. If A is maximal monotone $I = (I + A)^{-1} + (I + A^{-1})^{-1}$. Indeed if $x \in H$, $x = \xi + \eta$ with $\xi = J_1 x$ and $\eta = A_1 x$. $[\xi, \eta] \in A$ since

$A_1 \subseteq AJ_1$. Therefore $[\eta, \xi] \in A^{-1}$ which is maximal monotone. Here $\xi \in A^{-1}\eta$ and $x = \eta + \xi \in \eta + A^{-1}\eta = (I + A^{-1})(\eta)$. Therefore $\eta = (I + A^{-1})^{-1}(x)$, hence $x = (I+A)^{-1}x + (I + A^{-1})^{-1}x$. Since $\partial\varphi$ is maximal monotone, so is $\lambda\partial\varphi$ and $I = (I + \lambda\partial\varphi)^{-1} + (I + (\lambda\partial\varphi)^{-1})^{-1}$. But $I - (I + \lambda\partial\varphi)^{-1} = \lambda(\partial\varphi)_\lambda = (I + (\lambda\partial\varphi)^{-1})^{-1}$. Hence $\partial\varphi_\lambda = \lambda^{-1}(I + (\lambda\partial\varphi)^{-1})^{-1} = (\lambda I + (\lambda\partial\varphi)^{-1} \circ \lambda)^{-1} = (\lambda I + \partial\varphi^{-1})^{-1}$. But $\lambda I + \partial\varphi^{-1} = \partial[\lambda e + \varphi^*]$ and $(\lambda I + \partial\varphi^{-1})^{-1} = (\partial[\lambda e + \varphi^*])^{-1} = \partial[(\lambda e + \varphi^*)^*]$. Therefore $\varphi_\lambda = (\lambda e + \varphi^*)^* + C$ but $\varphi_\lambda(0) = 0$ and $(\lambda e + \varphi^*)^*(0) = 0$, thus $\varphi_\lambda = (\lambda e + \varphi^*)^*$.

Finally let us recall some property of convergence. If $x \in D(\partial\varphi)$, then $(\partial\varphi)_\lambda(x) \to (\partial\varphi)^\circ(x)$ as $\lambda \downarrow 0$ (where $\partial\varphi^\circ(x)$ is the element of $\partial\varphi(x)$ of minimal norm) and $|(\partial\varphi)_\lambda(x)| \leq |\partial\varphi^\circ(x)|$.

We conclude this section by stating a lemma which will be used later.

Lemma 2. Let $\psi: H \to [0, \infty]$ convex, lower semi continuous, with $\psi(0) = 0$. Let $\lambda_n \downarrow 0$ as $n \to \infty$, and let $(\alpha_{\lambda_n}, x_{\lambda_n}) \in \mathbb{R} \times H$ such that 1) $\alpha_{\lambda_n} \to \alpha$ 2) $x_{\lambda_n} \to x$ 3) $\alpha x \in \overline{D(\psi)}$ 4) $\psi'_{\lambda_n}(x_{\lambda_n}) = \alpha_{\lambda_n} x_{\lambda_n}$.

Then a) $\partial\psi(x) \ni \alpha x$

b) $\lim_{n\to\infty} \psi_{\lambda_n}(x_{\lambda_n})$ exists and is equal to $\psi(x)$.

Proof of Lemma 2

a) By the monotonicity of ψ'_{λ_n}, we have:

$$(\alpha_{\lambda_n} x_{\lambda_n} - \psi'_{\lambda_n}(v), x_{\lambda_n} - v) \geq 0 \quad \text{for all } v \in D(\partial\psi).$$

Hence $(\alpha x - (\partial\varphi)^\circ(v), x - v) \geq 0$ for all $v \in D(\partial\varphi)$. Since $\partial\varphi^\circ$ is a principal section of $\partial\varphi$, $x \in D(\partial\varphi)$ and $\alpha x \in \partial\varphi(x)$.

b) We have

$$\varphi_{\lambda_n}(x_{\lambda_n}) - \varphi_{\lambda_n}(x) \geq (\varphi'_{\lambda_n}(x), x_{\lambda_n} - x) = \varepsilon_n$$

$$\varphi_{\lambda_n}(x) - \varphi_{\lambda_n}(x_{\lambda_n}) \geq (\varphi'_{\lambda_n}(x_{\lambda_n}), x - x_{\lambda_n}) = \eta_n .$$

Hence $\varphi_{\lambda_n}(x) + \varepsilon_n \leq \psi_{\lambda_n}(x_{\lambda_n}) \leq \psi_{\lambda_n}(x) - \eta_n$. But $\psi_{\lambda_n}(x) \uparrow \psi(x) < \infty$, since $x \in D(\partial\varphi)$, and $\varepsilon_n, \eta_n \to 0$ since $x_{\lambda_n} \to x$, $\psi'_{\lambda_n}(x_{\lambda_n}) = \alpha_{\lambda_n} x_{\lambda_n}$ is bounded, as is $\varphi'_{\lambda_n}(x)$.

4. Proof of Theorem 1

a) Critical values

For $R \geq 0$, $k \in \mathbb{N}$, let $\gamma_k := \{\Gamma \subseteq \partial B_R \mid \Gamma \in \Sigma(H), \Gamma \text{ compact}, \gamma(\Gamma) \geq k\}$. $b_k := \inf_{\Gamma \in \gamma_k} \sup_{x \in \Gamma} \varphi(x)$. We claim that $b_k < \infty$, for each $k \in \mathbb{N}$, and each $R > 0$. First observe that for each $R > 0$, and each $k \in \mathbb{N}$, $\partial B_R \cap D(\varphi)$ contains a k-dimensional sphere Γ_k. If not, since $x \in D(\varphi) \cap \partial B_{R'}$ with $R' > R$ only if $x \in D(\varphi) \cap \partial B_R$, (by the convexity of $D(\varphi)$ and the fact that $0 \in D(\varphi)$), we would have $D(\varphi) \subseteq E_\ell$ where E_ℓ is a ℓ-dimensional subspace with $\ell < k$. But this would contradict assumption 3). Now let $R > 0$, $k \in \mathbb{N}$ given. We know that for $\varepsilon > 0$, there exists a k-dimensional sphere $\Gamma_k^{R+\varepsilon}$ contained in $\partial B_{R+\varepsilon} \cap D(\varphi)$. Let $E_k = \operatorname{span} \Gamma_k^{R+\varepsilon}$. E_k is a finite dimensional subspace. Let $\tilde{\varphi}$ the restriction of φ to E_k. Clearly $D(\tilde{\varphi}) \supseteq \Gamma_k^{R+\varepsilon}$ and $\tilde{\varphi}$ is continuous at each point x contained in the interior of Conv $\Gamma_k^{R+\varepsilon}$ (convex closure) relative to E_k, in particular on $\Gamma_k^R := \partial B_R \cap \text{Conv}\,\Gamma_k^{R+\varepsilon}$. Γ_k^R is a k-dimensional sphere, the genus of Γ_k^R is k, so $\Gamma_k^R \in \gamma_k$. Since Γ_k^R is compact and $\tilde{\varphi}$ is continuous on it, $\sup_{x \in \Gamma_k^R} \tilde{\varphi}(x) = \sup_{x \in \Gamma_k^R} \varphi(x) < \infty$, therefore $b_k = \inf_{\Gamma \in \gamma_k} \sup_{x \in \Gamma} \varphi(x) < \infty$.

$0 \le b_k \le b_{k+1}$ follows immediately from the definition.

For $R > 0$, $k \in \mathbb{N}$, $\lambda > 0$, let $b_k^\lambda := \inf_{\Gamma \in \gamma} \sup_{x \in \Gamma} \varphi_\lambda(x)$. Since $\varphi_\lambda \uparrow \varphi$ as $\lambda \downarrow 0$, we have $b_k^\lambda \le b_k^\mu$ if $\mu \le \lambda$ and $b_k^\lambda \le b_k$, for $\lambda > 0$. Therefore $\tilde{b}_k := \sup_{\lambda > 0} b_k^\lambda < \infty$. We claim that $\sup_{k \in \mathbb{N}} \tilde{b}_k = \infty$. If not, there exists $C > 0$, such that $b_k^\lambda < C$ for all $\lambda > 0$ and $k \in \mathbb{N}$. By the definition of b_k^λ, there exists $\Gamma_k \in \gamma_k$ such that $\sup_{x \in \Gamma_k} \varphi_{1/k}(x) \le C$ for all $k \in \mathbb{N}$; hence $\sup_{x \in \Gamma_k} \varphi(J_{1/k}x) \le C$ and $\sup_{x \in \Gamma_k} \frac{1}{2k} |A_{1/k}x|^2 \le C$. But $|J_{1/k}x| = |x - \frac{1}{k} A_{1/k}x| \ge R - \sqrt{2C}\, \frac{1}{\sqrt{k}}$, if $x \in \Gamma_k$. Therefore there exists k_0 such that for $k \ge k_0$, $x \in \Gamma_k$, $|J_{1/k}x| \ge \frac{R}{2}$. Let $\tilde{\Gamma}_k$ be the image of Γ_k by $J_{1/k}$. $\tilde{\Gamma}_k$ is compact and symmetric since Γ_k is compact, and symmetric and $J_{1/k}$ is continuous and odd. Since $0 \notin \tilde{\Gamma}_k$ for $k \ge k_0$, $\tilde{\Gamma}_k \in \Sigma(H)$. Since $J_{1/k}$ is continuous and odd, $\gamma(\tilde{\Gamma}_k) \ge \gamma(\Gamma_k) \ge k$ for $k \ge k_0$. Let $\bar{\Gamma} := \{x \in H \,|\, \varphi(x) \le C$ and $|x| \ge \frac{R}{2}\}$. $\bar{\Gamma}$ is not empty since $\tilde{\Gamma}_k \subseteq \bar{\Gamma}$ for $k \ge k_0$. $\bar{\Gamma}$ is compact by assumption 2), and symmettic since φ is even and $0 \notin \bar{\Gamma}$ by definition; therefore $\gamma(\bar{\Gamma}) = k_1 < \infty$. But, for $\bar{k} = \max(k_0, k_1) + 1$, $\tilde{\Gamma}_{\bar{k}} \subseteq \bar{\Gamma}$, hence $\gamma(\tilde{\Gamma}_{\bar{k}}) \le k_1$ and $\gamma(\tilde{\Gamma}_{\bar{k}}) \ge \bar{k} > k_1$, a contradiction. Thus $\sup_{k \in \mathbb{N}} \tilde{b}_k = \infty$.

b) Approximate solutions

Let $R > 0$, $k \in \mathbb{N}$. Let $\bar{\lambda} := \frac{R^2}{2b_k}$ if $b_k > 0$, and $\bar{\lambda} > 0$ arbitrary if $b_k = 0$. We claim that for $0 < \lambda < \bar{\lambda}$, there exists $(\mu_{k,\lambda}, u_{k,\lambda}) \in \mathbb{R} \times H$ such that:

$$|u_{k,\lambda}| = R \tag{4.1}$$

$$\varphi'_\lambda(u_{k,\lambda}) = \mu_{k,\lambda} u_{k,\lambda}, \qquad \mu_{k,\lambda} \geq 0 \tag{4.2}$$

$$\varphi_\lambda(u_{k,\lambda}) = b_k^\lambda . \tag{4.3}$$

Let $0 < \lambda < \bar{\lambda}$. Since $\varphi_\lambda \in C^1(H, \mathbb{R})$ and φ'_λ is Lipschitz continuous, by the Lemma 1 we are done provided that $\varphi_\lambda|_{\partial B_R}$ satisfies (P.S) at b_k^λ. Let $x_n \in \partial B_R$ such that $\varphi_\lambda(x_n) \to b_k^\lambda$ and $|\varphi'_\lambda(x_n) - (\varphi'_\lambda(x_n), x_n)R^{-2} x_n| \to 0$. We have $\varphi'_\lambda(x_n) = A_\lambda x_n$ and $x_n = J_\lambda x_n + \lambda A_\lambda x_n$. Therefore $(1 - \lambda(A_\lambda x_n, x_n)R^{-2})A_\lambda x_n - (A_\lambda x_n, x_n)R^{-2} J_\lambda x_n \to 0$. But $0 \leq \lambda(A_\lambda x_n, x_n)R^{-2} \leq 1$. We can extract a subsequence, still denoted by x_n, such that $\lambda(A_\lambda x_n, x_n)R^{-2} \to \alpha \in [0,1]$. We claim that $\alpha < 1$. Otherwise $J_\lambda x_n \to 0$ since $A_\lambda x_n$ is bounded and $(A_\lambda x_n, x_n)R^{-2}$ doesn't converge to 0. Then $\frac{\lambda}{2}|A_\lambda x_n|^2 = \frac{1}{2}(A_\lambda x_n, x_n) - \frac{1}{2}(J_\lambda x_n, A_\lambda x_n) \to \frac{R^2}{2\lambda}$. Hence by using the lower semi continuity of φ (Assumption 2), $0 = \varphi(0) \leq \underline{\lim}\, \varphi(J_\lambda x_n) = \underline{\lim}\, [\varphi_\lambda(x_n) - \frac{\lambda}{2}|A_\lambda x_n|^2] = b_k^\lambda - \frac{R^2}{2\lambda} < b_k^\lambda - \frac{R^2}{2\bar{\lambda}} \leq 0$, a contradiction. Therefore $\alpha < 1$. Since $\varphi_\lambda(x_n) = \frac{\lambda}{2}|A_\lambda x_n|^2 + \varphi(J_\lambda x_n)$ is bounded, $\varphi(J_\lambda x_n) \leq C$ and by Assumption 2, $J_\lambda x_n$ lies in a compact subset of H. We extract a subsequence still denoted by x_n, such that $J_\lambda x_n \to z$. Therefore $A_\lambda x_n$ converges strongly to $\frac{\alpha\lambda^{-1}}{1-\alpha} z$ and $x_n \to (1-\alpha)^{-1} z$. Thus

$\varphi_\lambda\big|_{\partial B_R}$ satisfies (P. S) at b_k^λ. Finally, observe that in (4.2), $\mu_{k,\lambda} \geq 0$ follows from the monotony of φ'_λ and the fact that $\varphi'_\lambda(0) = 0$.

c) Limit procedure

Let $R > 0$, $k \in \mathbb{N}$. We claim that there exists $\lambda_n \downarrow 0$ and $u_k \in H$ such that $u_{k,\lambda_n} \to u_k$. Let $0 < \lambda < \bar{\lambda}$. $\varphi_\lambda(u_{k,\lambda}) = b_k^\lambda \leq b_k < \infty$. Therefore $\varphi(J_\lambda u_{k,\lambda}) \leq b_k$ and $\frac{\lambda}{2}|A_\lambda u_{k,\lambda}|^2 \leq b_k$. By Assumption 2), there exists a sequence $\lambda_n \downarrow 0$ and $u_k \in H$ such that $J_{\lambda_n} u_{k,\lambda_n} \to u_k$. $\lambda|A_\lambda u_{k,\lambda}|^2 \leq 2b_k$ implies that $\lambda_n A_{\lambda_n} u_{k,\lambda_n} \to 0$, hence $u_{k,\lambda_n} = J_{\lambda_n} u_{k,\lambda_n} + \lambda_n A_{\lambda_n} u_{k,\lambda_n} \to u_k$. In particular $|u_k| = R$.

We shall prove that μ_{k,λ_n} is bounded. We already know that $\mu_{k,\lambda_n} \geq 0$. Assume that there exists a subsequence $\lambda_{n_j} \downarrow 0$ such that $\mu_{k,\lambda_{n_j}} \to +\infty$. For n_j big enough, $\mu_{k,\lambda_{n_j}} > 0$. We have for all $v \in D(\partial\varphi)$:

$$\left(\frac{1}{\mu_{k,\lambda_{n_j}}} A_{\lambda_{n_j}} u_{k,\lambda_{n_j}} - \frac{1}{\mu_{k,\lambda_{n_j}}} A_{\lambda_{n_j}} v,\ u_{k,\lambda_{n_j}} - v\right) \geq 0$$

by the monotonicity of A_λ. Since $\frac{1}{\mu_{k,\lambda_{n_j}}} A_{\lambda_{n_j}} v$, $u_{k,\lambda_{n_j}}$ $u_{k,\lambda_{n_j}} =$

$u_{k,\lambda_{n_j}} \to u_k$, $\frac{1}{\mu_{k,\lambda_{n_j}}} A_{\lambda_{n_j}} v \to 0$ ($A_\lambda v$ is bounded since $v \in D(\partial\varphi)$). Therefore we get:

$$(u_k, u_k - v) \geq 0 \text{ for all } v \in \overline{D(\partial\varphi)} = \overline{D(\varphi)} = H$$

by Assumption 3. But $u_k \neq 0$, a contradiction.

We know that $u_{k,\lambda_n} \to u_k$, $|\mu_{k,\lambda_n}| \leq C$, therefore we can extract a subsequence, still denoted by λ_n, such that $u_{k,\lambda_n} \to u_k$, $\mu_{k,\lambda_n} \to \mu_k \geq 0$.

We can apply Lemma 2, with $\psi = \varphi$, $\alpha_{\lambda_n} = \mu_{k,\lambda_n}$, $x_{\lambda_n} = u_{k,\lambda_n}$. Observe that $\mu_k u_k \in \overline{D(\varphi)} = H$. Therefore we have $|u_k| = R$, $\partial\varphi(u_k) \ni \mu_k u_k$ and $\varphi(u_k) = \lim_{\lambda_n \downarrow 0} \varphi_{\lambda_n}(u_{k,\lambda_n}) = \lim_{\lambda_n \downarrow 0} b_k^{\lambda_n} = \sup_{\lambda > 0} b_k^{\lambda} = \tilde{b}_k$. But in the first part of the proof, we showed that $\sup_{k \in \mathbb{N}} \tilde{b}_k = \infty$. Hence $\sup_{k \in \mathbb{N}} \varphi(u_k) = \sup_{k \in \mathbb{N}} \tilde{b}_k = \infty$ and we are done.

5. Proof of Theorem 2

a) We claim that if $\psi: H \to [0, \infty]$ satisfies the hypotheses of Theorem 2, then $D(\psi^*) = H$, ψ^* is weakly continuous, $\psi^*(x) = 0$ if and only if $x = 0$, $\partial\psi^*(x) \ni 0$ iff $x = 0$ and ψ^* is even. ψ is convex, even, lower semi continuous by Assumption 2, $\not\equiv +\infty$ by Assumption 1, and $\partial\psi^* = (\partial\psi)^{-1}$ is defined everywhere and maps bounded sets into bounded sets by Assumption 3. Then $D(\psi^*) = H$. ψ^* is convex lower semi continuous and by Hahn-Banach, weakly lower semi continuous. We prove that if $x_n \rightharpoonup x$, then $\overline{\lim}\, \psi^*(x_n) \leq \psi^*(x)$. By the surjectivity of $\partial\psi$, there exists $y_n \in D(\partial\psi)$ with $x_n \in \partial\psi(y_n)$. Since x_n is bounded, so is y_n and $\psi(y_n) \leq \psi(y_n) + \psi^*(x_n) = (x_n, y_n) \leq C$.

By Assumption 2), y_n lies in a compact subset of H, therefore $\lim_{n\to\infty} (y_n, x_n - x) = 0$. But $\psi^*(x_n) \leq (y_n, x_n - x) + \psi^*(x)$, hence $\overline{\lim}\, \psi^*(x_n) \leq \psi^*(x)$. Therefore ψ^* is weakly continuous. Clearly $\psi^*(0) = 0$. Assume $\psi^*(x) = 0$. Then $(x, y) \leq \psi(y)$ for all $y \in H$, hence $x \in \partial\psi(0)$, so $x = 0$ by Assumption 1. Clearly $\partial\psi^*(0) \ni 0$. Assume $\partial\psi^*(x) \ni 0$, then $x \in \partial\psi(0)$, hence $x = 0$.

Finally, it is clear that φ^* is even, since $\varphi^*(-x) = \sup_{y \in H} \{(-x, y) - \varphi(y)\} =$

$\sup_{-y \in H} \{(x, y) - \varphi(-y)\} = \sup_{-y \in H} \{(x, y) - \varphi(y)\} = \varphi^*(x)$.

Observe that for all $\lambda > 0$, $\lambda e + \varphi$ satisfies the assumptions of Theorem 2. $(e(x) := \frac{1}{2}|x|^2)$. Therefore for all $\lambda > 0$, φ and $(\varphi^*)_\lambda =$ $(\lambda e + \varphi^{***}) = (\lambda e + \varphi)^*$ are weakly continuous, ≥ 0, $= 0$ iff $x = 0$, $\partial\varphi^*(x)$ and $\partial(\varphi^*)_\lambda(x) = 0$ iff $x = 0$, and φ^*, $(\varphi^*)_\lambda$ are even. Therefore, by a result stated in Section 2, $c_k := \sup_{\Gamma \in \gamma_k} \min_{x \in \Gamma} \varphi^*(x)$ and $c_k^\lambda := \sup_{\Gamma \in \gamma_k} (\varphi^*)_\lambda(x)$ satisfy $\infty > c_k$, $c_k^\lambda > 0$ and $\inf_{k \in \mathbb{N}} c_k = 0$. Since $c_k^\lambda \leq c_k$, $\tilde{c}_k := \sup_{\lambda > 0} c_k^\lambda$, satisfies $\tilde{c}_k > 0$ and $\inf_{k \in \mathbb{N}} \tilde{c}_k = 0$.

b) We claim that for any $R > 0$, $k \in \mathbb{N}$ and $\lambda > 0$, there exists $v_{k,\lambda} \in H$, $\nu_{k,\lambda} > 0$ such that

$$|v_{k,\lambda}| = R \tag{5.1}$$

$$(\varphi^*)'_\lambda(v_{k,\lambda}) = \nu_{k,\lambda} v_{k,\lambda} \tag{5.2}$$

$$(\varphi^*)_\lambda(v_{k,\lambda}) = c_k^\lambda . \tag{5.3}$$

First observe that if $(\varphi^*)'_\lambda(v_{k,\lambda}) = \nu_{k,\lambda} v_{k,\lambda}$ for some $\nu_{k,\lambda} \in \mathbb{R}$, then $\nu_{k,\lambda}$ has to be > 0 . Indeed, $\nu_{k,\lambda}$ is ≥ 0 by the monotonicity of $(\varphi^*)'_\lambda$ and the fact that $(\varphi^*_\lambda)'(0) = 0$. It is > 0, since $(\varphi^*)'_\lambda(x) = 0$ iff $x = 0$, and $v_{k,\lambda} \neq 0$. Now (5.1) - (5.3) will be proved, if $(\varphi^*)_\lambda\big|_{\partial B_R}$ satisfies the (P.S) condition at c_k^λ, by Lemma 1. Let $x_n \in \partial B_R$, such that

$(\varphi^*)_\lambda(x_n) \to c_k^\lambda > 0$ and $|(\varphi^*)'_\lambda(x_n) - ((\varphi^*)'_\lambda(x_n), x_n)R^{-2} x_n| \to 0$. Since $x_n \in \partial B_R$, we can extract a subsequence, still denoted by x_n, such that $x_n \to x \in H$. We claim that $(\varphi^*)'_\lambda(x_n) \to (\varphi^*_\lambda)'(x)$, at least for a subsequence. First observe that $(\varphi^*)_\lambda = (\lambda e + \varphi)^*$, hence $(\varphi^*)'_\lambda = \partial[(\varphi^*)_\lambda] = \partial[(\lambda e + \varphi)^*] = (\partial[\lambda e + \varphi])^{-1} = (\lambda I + \partial\varphi)^{-1}$. Since $\lim\limits_{\substack{|x| \to \infty \\ x \in D(\varphi)}} \dfrac{\lambda/2|x|^2 + \varphi(x)}{|x|} = \infty$, $(\lambda I + \partial\varphi)^{-1}$ maps bounded sets, into bounded sets. In particular $y_n := (\lambda I + \partial\varphi)^{-1} x_n$ is bounded. But $x_n - \lambda y_n \in \partial\varphi(y_n)$, therefore $\varphi(y_n) \leq \varphi(y_n) + \varphi^*(x_n - \lambda y_n) = (x_n - \lambda y_n, y_n) \leq (x_n, y_n) \leq C$ for some $C > 0$. By Assumption 2), we can extract subsequences, still denoted by x_n and y_n, such that $y_n \to y \in H$. Now $(y_n - (\lambda I + \partial\varphi)^{-1} v, x_n - v) \geq 0$ for all $v \in H$, therefore, $(y - (\lambda I + \partial\varphi)^{-1} v, x - v) \geq 0$ for all $v \in H$ and $y = (\lambda I + \partial\varphi)^{-1} x$ by the maximal monotony of $(\lambda I + \partial\varphi)^{-1}$. So $(\varphi^*)'_\lambda(x_n) \to (\varphi^*)'_\lambda(x)$ and $x_n \to R^2[((\varphi^*)'_\lambda(x), x)]^{-1}(\varphi^*)'_\lambda(x)$ since $((\varphi^*)'_\lambda(x), x) > \varphi^*_\lambda(x) > 0$. Therefore $(\varphi^*)_\lambda\big|_{\partial B_R}$ satisfies the (P.S) condition at c_k^λ.

c) We claim that for each $R > 0$, each $k \in \mathbb{N}$, $\nu_{k,\lambda}$ is bounded as $\lambda \downarrow 0$. We have $(\varphi^*)'_\lambda = \partial(\varphi^*)_\lambda = (\partial\varphi^*)_\lambda = ((\partial\varphi)^{-1})_\lambda$ and $|(\partial\varphi)^{-1}_\lambda(v_{k,\lambda})| \leq |(\partial\varphi^{-1})^\circ(v_{k,\lambda})|$. But $(\partial\varphi)^{-1}$ maps bounded sets into bounded sets, so $|\nu_{k,\lambda}| = R^{-1}|(\varphi^*)'_\lambda(v_{k,\lambda})| \leq R^{-1}|(\partial\varphi^{-1})^\circ(v_{k,\lambda})| \leq C$ for some $C > 0$. So $((\varphi^*)_\lambda)^*(\nu_{k,\lambda}) \leq ((\varphi^*)_\lambda)^*(\nu_{k,\lambda}) + (\varphi^*)_\lambda(v_{k,\lambda}) = (v_{k,\lambda}, \nu_{k,\lambda}) \leq CR^2$. But $((\varphi^*)_\lambda)^* = \lambda e + \varphi$, so $\varphi(\nu_{k,\lambda}) \leq CR^2$. Therefore $\nu_{k,\lambda}$ lies in a compact set of H. We can extract a subsequence $\lambda_n \downarrow 0$ such that $v_{k,\lambda_n} \to v_k$ and $\nu_{k,\lambda_n} \to \nu_k$. Since $\nu_k\, v_k \in H = \overline{D(\varphi^*)}$, by Lemma 2, we get: $|v_k| = R$.

$\partial\varphi^*(v_k) \ni \nu_k v_k$ and $\varphi^*(v_k) = \lim_{\lambda_n \downarrow 0} (\varphi^*)_{\lambda_n}(v_{k,\lambda_n}) = \sup_{\lambda>0} c_k^\lambda = \tilde{c}_k$. $\partial\varphi^*(v_k) = (\partial\varphi)^{-1}(v_k) \ni \nu_k v_k$. Again $\nu_k \geq 0$ and even > 0, otherwise $v_k \in \partial\varphi(0)$, which is impossible by Assumption 1 and the fact that $|v_k| = R$. Moreover we already know that $\inf_{k\in\mathbb{N}} \tilde{c}_k = 0$, so $\inf_{k\in\mathbb{N}} \varphi^*(v_k) = 0$.

6. Proof of Theorem 3

a) For $R > 0$, let $b := \inf_{|x|=R} \varphi(x)$. $\partial B_R \cap D(\varphi)$ is not empty, otherwise $D(\varphi) \subseteq \overline{B}_R$, which contradicts Assumption 3. Thus $0 \leq b < \infty$. For $\lambda > 0$, let $b^\lambda := \inf_{|x|=R} \varphi_\lambda(x)$. $0 \leq b^\lambda \leq b$.

b) Let $R > 0$, $\overline{\lambda} := \frac{R^2}{2b}$ if $b \neq 0$ and arbitrary positive if $b = 0$. If $0 < \lambda < \overline{\lambda}$, $\varphi_\lambda\big|_{\partial B_R}$ satisfies (P.S) at b, as in the proof of Theorem 1. Then there exists $u_\lambda \in H$, $\mu_\lambda \geq 0$ such that i) $|u_\lambda| = R$ ii) $\varphi_\lambda'(u_\lambda) = \mu_\lambda u_\lambda$ iii) $\varphi_\lambda(u_\lambda) = b_\lambda$.

c) We can apply the same proof as in Theorem 1, to get the existence of a sequence $\lambda_n \downarrow 0$ such that $u_{\lambda_n} \to u$ and $\mu_{\lambda_n} \to \mu \geq 0$. Observe that we get $(u, u-v) \geq 0$ for all $v \in \overline{D(\partial\varphi)} = \overline{D(\varphi)} = P$. Since P is a cone, we can choose $v = 2u$ and we get the same contradiction as earlier which proves the boundedness of μ_{λ_n}. So $\mu_{\lambda_n} \to u$, $\mu_{\lambda_n} \to \mu$ and $\mu u \in P$ since $\mu \geq 0$ and $u \in P$. Therefore $\mu u \in \overline{D(\varphi)}$ and we can invoke Lemma 2, to get $\partial\varphi(u) \ni \mu u$ and $\varphi(u) = \lim_{\lambda_n \downarrow 0} \varphi_{\lambda_n}(u_{\lambda_n})$. But $\varphi(u) = \lim_{\lambda_n \downarrow 0} \varphi_{\lambda_n}(u_{\lambda_n}) = \sup_{\lambda>0} b^\lambda \leq b = \inf_{|v|=R} \varphi(v)$. Since $|u| = R$, $\varphi(u) = b$ and this concludes the proof of Theorem 3.

7. Proof of Theorem 4

Let $R > 0$. Then $\overline{B}_R$ is weakly compact and φ is weakly continuous. Therefore there exists $u \in \overline{B}_R$ such that $\varphi(u) = \max_{|v| \leq R} \varphi(v)$. Since $\overline{B}_R$ and φ are convex, $u \in \partial B_R$ or can be chosen in ∂B_R if there is more than one maximum. Since $D(\varphi) = H$, $D(\partial\varphi) = H$. Therefore for all $z \in \partial B_R$, $0 \geq \varphi(z) - \varphi(u) \geq (y, z-u)$, for $y \in \partial\varphi(u)$. If $0 \in \partial\varphi(u)$, we are done. If not, let $y \in \partial\varphi(u)$, $y \neq 0$. Then $(\frac{Ry}{|y|}, z-u) \leq 0$, for all $z \in \partial B_R$. By taking $z = \frac{Ry}{|y|}$, we get $R^2 \leq (\frac{Ry}{|y|}, u)$ hence $\frac{Ry}{|y|} = u$ and $\partial\varphi(u) \ni \lambda u$, for some $\lambda \geq 0$.

8. An example

Let $\Omega \subseteq \mathbb{R}^n$ a bounded domain with smooth boundary. Let $\beta \subseteq \mathbb{R} \times \mathbb{R}$ an odd maximal monotone graph with $0 \in \beta(0)$ and $D(\beta) = \mathbb{R}$. Let $j: \mathbb{R} \to \mathbb{R}$ the unique convex function such that $\beta = \partial j$ and $j(0) = 0$.

Let $H = L^2(\Omega)$ and $\varphi: H \to [0, \infty]$ defined by $\varphi(u) := \frac{1}{2}\int_\Omega \operatorname{grad}^2 u \, dx + \int_\Omega j(u)dx$ if $u \in \mathring{W}^{1,2}(\Omega)$ and $j(u) \in L^1(\Omega)$, $+\infty$ otherwise.

It is well-known [see for example [8]] that φ is convex, even, lower semi continuous, $D(\partial\varphi) = \mathring{W}^{1,2}(\Omega) \cap W^{2,2}(\Omega) \cap \{u \in L^2(\Omega) | \beta(u) \in L^2(\Omega)\}$ and $\partial\varphi(u) = -\Delta u + \beta(u)$. Since the injection of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact, $\{u \in L^2(\Omega) | \varphi(u) \leq c\}$ is compact in $L^2(\Omega)$ for all $c \geq 0$.

Clearly $D(\varphi)$ is dense in H. Therefore φ satisfies the hypothesis of Theorem 1 and for all $R > 0$, there exists infinitely many distinct pairs of solutions of

$$-\Delta u + \beta(u) \ni \lambda u, \int_\Omega |u|^2 dx = R^2, \ u \in \mathring{W}^{1,2}(\Omega) \cap W^{2,2}(\Omega) \text{ and } \beta(u) \in L^2(\Omega). \tag{*}$$

By a standard regularity result $u \in C^{1,\alpha}(\Omega)$ for $\alpha \in]0,1[$. If β is univalued and belongs to $C^1(\mathbb{R})$, $u \in C^{2,\alpha}(\Omega)$ and therefore u is a classical solution of (*).

Clearly φ satisfies the condition $\lim\limits_{\substack{|u| \to \infty \\ u \in D(\varphi)}} \frac{\varphi(u)}{|u|} = +\infty$. If we assume moreover that $0 = \beta(0)$, then $0 = \partial\varphi(0)$. By the Theorem 2, we get the existence of infinitely many distinct pairs of solutions for (*) satisfying $\lambda^2 \int_\Omega |u|^2 dx = R^2$, $u \in \mathring{W}^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ with $\beta(u) \in L^2(\Omega)$.

As an application of Theorem 3, we shall give the following lemma.

<u>Lemma 3.</u> Let H a real ordered Hilbert space with a positive cone P satisfying $(u, v) \geq 0$ if $u, v \in P$ and such that for all $z \in P$, there exists $z^+, z^- \in P$ with $z = z^+ - z^-$, $(z^+, z^-) = 0$. Let $\varphi: H \to [0, \infty]$ convex satisfying

$$\varphi(0) = 0 \tag{1}$$

$$\{x \in H \mid \varphi(x) \leq c\} \text{ is compact for } c \geq 0 \tag{2}$$

$$\overline{D(\varphi)} \supseteq P \tag{3}$$

$$(I + \lambda\,\partial\varphi)^{-1} P \subseteq P \text{ for all } \lambda > 0\,. \tag{4}$$

Then for all $R > 0$, there exists $u \in H$, $\lambda \geq 0$ with

a) $|u| = R$

b) $\partial\varphi(u) \ni \lambda u$

c) $u \in P$.

Proof of Lemma 3

Let $\tilde{\varphi} := \varphi + I_P$ where $I_P: H \to [0, \infty]$ is defined by $I_P(u) = 0$ if $u \in P$ and $+\infty$ if $u \notin P$. By Proposition 4.5 of [1], $\partial\tilde{\varphi} = \partial\varphi + \partial I_P$ and $\overline{D(\tilde{\varphi})} = P$. Clearly $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}$ satisfies Assumption 2 of Theorem 3, so for all $R > 0$, there exists $\lambda \geq 0$ and $\bar{u} \in \partial B_R \cap P$ with $\tilde{\varphi}(\bar{u}) = \inf_{|v|=R} \tilde{\varphi}(v)$ and $\partial\tilde{\varphi}(\bar{u}) \ni \lambda\bar{u}$. Let $\bar{w} \in \partial I_P(\bar{u})$ such that $\partial\varphi(\bar{u}) + \bar{w} \ni \lambda\bar{u}$. We have

$\partial I_P(\bar{u}) = \{z \in H | (z, v-\bar{u}) \leq 0$ for all $v \in P\}$. We claim that if $z \in \partial I_P(\bar{u})$, $z \in -P$ and $(z, \bar{u}) = 0$. Indeed, by assumption there exists z^+ and $z^- \in P$ such that $z = z^+ - z^-$ and $(z^+, z^-) = 0$. Hence for all $\alpha \geq 0$ we have $\alpha|z^+|^2 - (z, \bar{u}) \leq 0$. This is possible only if $z^+ = 0$. So $z \in -P$. By taking $v = \frac{1}{2}\bar{u}$, we get $\frac{1}{2}((-z), -\bar{u}) \geq 0$, hence $(z, \bar{u}) = 0$. So we have $\bar{u} \in P$ and $\bar{z} = -\bar{w} \in P$ such that i) $\bar{u} + \partial\varphi(\bar{u}) \ni (\lambda+1)\bar{u} + \bar{z}$, ii) $(\bar{u}, \bar{z}) = 0$. Let $\psi(u) = \frac{1}{2}|u|^2 + \varphi(u) - (\lambda+1)(\bar{u}, u)$. From i) and ii) we get $\psi(\bar{u}) = \psi(\bar{u}) - (\bar{z}, \bar{u}) \leq \psi(u) - (\bar{z}, u)$ for all $u \in H$. Since $\bar{z} \in P$, $\psi(\bar{u}) \leq \psi(u)$ for all $u \in P$. Since $\bar{u} \in P$, $\psi(\bar{u}) = \operatorname{Inf}_{u \in P} \psi(u)$. But $\operatorname{Inf}_{u \in P} \psi(u) = \operatorname{Inf}_{u \in H} \psi(u)$. Indeed, there exists $\bar{\bar{u}} \in P$ such that $\psi(\bar{\bar{u}}) \leq \psi(u)$ for all $u \in H$. Such $\bar{\bar{u}}$ is unique and defined by $\bar{\bar{u}} = (I + \partial\varphi)^{-1}(\lambda+1)\bar{u}$. By Assumption 4 and since $(\lambda+1)\bar{u} \in P$, $\bar{\bar{u}} \in P$. Consequently, by uniqueness, $\bar{u} = \bar{\bar{u}}$ and $\bar{u} = (I + \partial\varphi)^{-1}(\lambda+1)\bar{u}$ or $\partial\varphi(\bar{u}) \ni \lambda\bar{u}$. This concludes the proof of Lemma 3.

As another example we consider again the equation

$$(*) \qquad -\Delta u + \beta(u) \ni \lambda u, \quad |u| = R. \quad u \in W^{2,2} \cap \overset{\circ}{W}{}^{1,2}.$$

We already mentioned that Assumptions 1), 2), 3) of Lemma 3 are satisfied. It is a standard result that the solution of the equation $u - \lambda \Delta u + \lambda \beta(u) = f$ belongs to L^2_+ if $\lambda > 0$ and $f \in L^2_+$. So the lemma can be applied and for all $R > 0$, (*) possesses a positive solution, which is $C^{1,\alpha}(\Omega)$ by a standard regularity argument. Moreover if β is univalued and C^1, $u \in C^{2,\alpha}(\Omega)$ and $u(x) > 0$ for $x \in \Omega$.

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In this paper, we prove Lyusternik-Schnirelmann type theorems for convex functions in a Hilbert space. We consider the problems $\partial\varphi(u) \ni \lambda u$ (and $(\partial\varphi)^{-1}(u) \ni \lambda u$) where $\partial\varphi$ is the subdifferential of an even convex function φ in a Hilbert space. We give conditions on φ for there to exist infinitely many distinct pairs of solutions having a prescribed norm.

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